Quantum mechanics I : tutorial solutions

2021.09.30 / exercises week 1

1. Through what potential difference must an electron be accelerated in order to have

(a) the same wavelength as an X-ray of wavelength 0.15 nm; and

(b) the same energy as the X-ray in part (a)?

a) the work done to move an electron through a potential V is $E = eV$

All this energy will transfer to the electron as kinetic energy $E_k = \frac{1}{2} m_e v^2$

We want to equate these and see what is the final potential needed.

Using the De Broglie relation

$$p = h/\lambda \longrightarrow v = \frac{h}{m_e \lambda}$$

Since we want $eV = \frac{1}{2} m_e v^2$

$$V = \frac{1}{2} \frac{h^2}{m_e \lambda^2 e} = \frac{(6.62 \cdot 10^{-34})^2}{2 \cdot (9.11 \cdot 10^{-31})(0.15 \cdot 10^{-9})^2 (1.6 \cdot 10^{-19})}$$

$$= 66.8 \text{ V}$$

where $h = 6.62 \cdot 10^{-34} \text{ m}^2 \text{ kg/s}$

$m_e = 9.11 \cdot 10^{-31} \text{ kg}$

$e = 1.6 \cdot 10^{-19} \text{ C}$

b) the energy of x-ray:

$$E = \frac{hc}{\lambda} = \frac{6.62 \cdot 10^{-34} \cdot 3 \cdot 10^{8}}{0.15 \cdot 10^{-9}} = 1.32 \cdot 10^{-15} \text{ J}$$

$$V = \frac{1.32 \cdot 10^{-15}}{1.6 \cdot 10^{-19}} = 8.25 \cdot 10^{3} \text{ V}$$

2. A Neodymium laser operates at a wavelength of 1.06×10^{-6} m. If the laser is operated in pulsed mode, emitting pulses of duration 3×10^{-11} s, what is the minimum spread in

(a) frequency and

(b) wavelength of the laser beam?

a) Since the pulses have duration $\Delta t = 3 \cdot 10^{-11}$ s, we have this duration to measure the energy of the laser.

Given the Heisenberg uncertainty principle $\Delta E \Delta t \geq \hbar$, the uncertainty on the energy will be

$$\Delta E \geq \frac{\hbar}{2} \frac{1}{\Delta t}$$

that in our case is

$$\Delta E \geq \frac{6.62 \cdot 10^{-34}}{2 \cdot 2\pi \cdot 3 \cdot 10^{-11}} = 1.75 \cdot 10^{-24} \text{ J}$$

Since $E = h\nu$

$$\Delta \nu = \frac{\Delta E}{h} = \frac{1.75 \cdot 10^{-24}}{6.62 \cdot 10^{-34}} = 2.64 \cdot 10^{9} \text{ Hz}$$

b) $$\Delta \lambda = c \frac{\Delta \nu}{\left(\frac{c}{\lambda}\right)^2 - \Delta \nu^2} = 3 \cdot 10^{8} \cdot \frac{2.64 \cdot 10^{9}}{\left(\frac{3 \cdot 10^{8}}{1.06 \cdot 10^{-6}}\right)^2 - \left(2.64 \cdot 10^{9}\right)^2}$$

$$= 3 \cdot 10^{8} \cdot \frac{2.64 \cdot 10^{9}}{8 \cdot 10^{28}}$$

$$= 9.9 \cdot 10^{-12} \text{ m}$$

3. When light of variable wavelength shines on a particular metal, no photoelectrons are emitted if the wavelength is greater than 550 nm. For what wavelength of light would the maximum kinetic energy of the photoelectrons be 3.5 eV?

3) threshold : 550 nm

All energy is transfered to electron - threshold

$$E_k = 3.5\ eV \cdot 1.6 \cdot 10^{-19}\ J/eV = 5.6 \cdot 10^{-19}\ J$$

the energy of the light has to be $E_k + E_{threshold}$

the energy corresponding to 550 nm is

$$E_{threshold} = \frac{hc}{\lambda} = \frac{6.62 \cdot 10^{-34} \cdot 3 \cdot 10^{8}}{550 \cdot 10^{-9}} = 3.6 \cdot 10^{-19}\ J$$

$$E_{light} = E_k + E_{threshold} = (5.6 + 3.6) \cdot 10^{-19}\ J$$
$$= 9.2 \cdot 10^{-19}\ J$$

The wavelength associated with it is:

$$\lambda = \frac{hc}{E_{light}} = \frac{6.62 \cdot 10^{-34} \cdot 3 \cdot 10^{8}}{9.2 \cdot 10^{-19}\ J} = 216\ nm$$

4. What is the typical de Broglie wavelength associated with an atom of helium in a gas at room temperature?

4) De Broglie relation $p = h/\lambda$

to find the λ associated with a certain momentum p, we need to find its velocity at room temperature.

$T = 293.15 \ K$

The equipartition theorem for a mono-atomic ideal gas states that the avg. kinetic energy of every atom is

$$\frac{1}{2} m v^2 = \frac{3}{2} k_B T \qquad \text{(in 3-D)}$$

where $k_B = 1.38 \cdot 10^{-23} \ \frac{J}{K}$

so

$$v = \sqrt{\frac{3 k_B T}{m}} = \sqrt{\frac{3 \cdot 1.38 \cdot 10^{-23} \cdot 293.15}{6.64 \cdot 10^{-27}}}$$

$$= 1.35 \cdot 10^{3} \ \frac{m}{s}$$

$$\lambda = \frac{h}{m v} = \frac{6.62 \cdot 10^{-34}}{6.64 \cdot 10^{-27} \cdot 1.35 \cdot 10^{3}} = 7.38 \cdot 10^{-11} \ m$$

5. Show that the real wave functions $\Psi_f = \sin(kx - \omega t)$ and $\Psi_f = \cos(kx - \omega t)$ are not solutions of the free-particle Schrödinger equation, whereas the complex wave function $\Psi_f = \exp(i(kx - \omega t))$ is. This is an important difference between classical waves and quantum mechanical wave functions.

5) free particle Schrödinger eq.

free particle → NO POTENTIAL

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi(x,t) + \underbrace{\cancel{V(x,t)\Psi(x,t)}}$$

$$i\hbar \frac{\partial}{\partial t} \sin(kx - wt) = -i w \hbar \cos(kx - wt)$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \sin(kx - wt) = \frac{\hbar^2 k^2}{2m} \sin(kx - wt)$$

In case of $\Psi_f(x,t) = e^{i(kx - wt)}$

$$i\hbar \frac{\partial}{\partial t} e^{i(kx - wt)} = \hbar w \, e^{i(kx - wt)}$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} e^{i(kx - wt)} = \frac{\hbar^2 k^2}{2m} e^{i(kx - wt)}$$

$\hbar w = \frac{\hbar^2 k^2}{2m}$ $\quad e^{i(kx - wt)}$ is SOLUTION

For non-relativistic matter waves $k = \frac{p}{\hbar}$

$$\rightarrow \quad E = \hbar w = \frac{p^2}{2m}$$

Just the dispersion relation ENERGY - MOMENTUM of a free particle

6. Consider the measurement of the position of a particle on the screen in a double-slit experiment. Describe what would happen

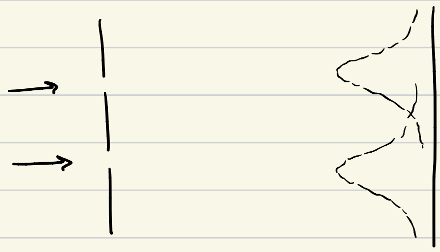
(a) on the basis of a classical particle picture;

(b) on the basis of a classical wave picture;

(c) on the basis of quantum mechanics.

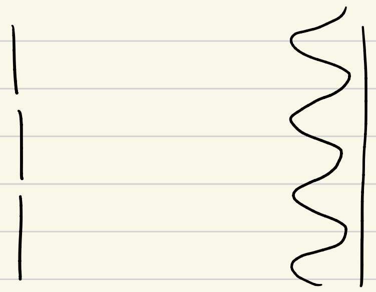
See lectures for complete explaination

a)



2 Gaussians

b)



Diffraction pattern

c) If it's observed where the q. particle passes through, it behaves like a classical particle

If it's observed the result on the screen only, it behaves like a wave.

TIPS

- de Broglie relation : $p = \frac{h}{\lambda}$

- uncertainty relation : $\Delta x \, \Delta p \geq \frac{\hbar}{2}$, $\Delta E \, \Delta t \geq \frac{\hbar}{2}$

  etc for all conjugated variables

- $E = \hbar \omega = h \nu = \frac{hc}{\lambda}$ and $p = \hbar k$

- The equipartition theorem for a mono-atomic ideal gas states that the avg. kinetic energy of every atom is

$$\frac{1}{2} m v^2 = \frac{3}{2} k_B T \qquad (\text{in 3-D})$$

- Probability of finding a particle between the position x and $x + dx$

  if $\psi(x,t)$ is the wave f^n of the particle

$$P(x, x+dx) = \int_x^{x+dx} dx' \, \underbrace{|\psi(x')|^2}$$

probability density