Maxwell-Boltzmann distribution

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1 Question 1

In one dimension, the velocity distribution function is

$$g\left(v_x\right) = \sqrt{\frac{m}{2\pi k_B T}} \exp\left(-\frac{m v_x^2}{2 k_B T}\right) .$$

a) Show that this is correctly normalised.

b) Show that

$$\langle v_x \rangle = 0 \qquad \text{and} \qquad \left\langle v_x^2 \right\rangle = \frac{k_B T}{m} .$$

2 Question 2

The Maxwell-Boltzmann energy distribution for an ideal gas is given by

$$p(\epsilon) d\epsilon = \frac{2}{\sqrt{\pi}} \beta^{3/2} \epsilon^{1/2} \exp(-\beta\epsilon) d\epsilon$$

where β is a constant.

a) What is the normalisation condition for a probability density function?

b) Using the standard integral

$$\int_0^\infty \sqrt{x} e^{-x} dx = \frac{\sqrt{\pi}}{2} ,$$

show that the function $p(\epsilon)$ is properly normalised.

c) Using the standard integral

$$\int_0^\infty x^{3/2} e^{-x} dx = \frac{3\sqrt{\pi}}{4} ,$$

and the equipartition theorem, demonstrate that $\beta = 1/\left(k_B T\right)$.

3 Question 3

The Maxwell velocity distribution for an ideal two-dimensional (2D) gas is expressed in terms of the probability density functions for each Cartesian component of the velocity vector $\vec{v}$, i.e. $p_x(v_x)$ where v_x is the x component of velocity.

a) State the meaning of the quantity $p_x(v_x)\,dv_x$. Assume that the probability density function $p_x(v_x)$ is a Gaussian function, i.e. $p_x(v_x) \propto \exp\left(-av_x^2\right)$, where a is a constant.

b) Show that for an ideal two-dimensional (2D) gas, in which particles are confined to the (x, y) plane, the speed distribution is given by

$$p(v)dv = fv\exp\left(-av^2\right)dv$$

where $v = \sqrt{v_x^2 + v_y^2} = |\vec{v}|$ is the particle speed and f is a constant.

c) Find f by requiring that $p(v)$ is normalised.

d) Comment why the prefactor in the equation above is proportional to v, while for the 3D case it is proportional to v^2.

e) Using the relation between the particle's speed and its energy, show that the energy probability density $p(E)$ for an ideal two-dimensional (2D) gas is

$$p(E)dE = \frac{2a}{m}\exp\left(-\frac{2a}{m}E\right)dE$$

where E is the kinetic energy of a gas particle with mass m.

f) Use $p(E)$ to find the average energy $\langle E \rangle$ of a particle in an ideal 2D gas.

g) Use your result together with the equipartition theorem, valid for systems in equilibrium at temperature T, to give an expression for the constant a.

h) Calculate the probability that the energy of a randomly chosen gas particle is less than k_BT. Give a numerical value.

These integrals may be useful for some calculations.

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}dx = \sqrt{2\pi\sigma^2}$$

$$\int_0^{\infty} \sqrt{x}e^{-x}dx = \frac{\sqrt{\pi}}{2}$$

$$\int_0^{\infty} x^2e^{-x^2}dx = \frac{\sqrt{\pi}}{4}$$

$$\int_0^{\infty} xe^{-x}dx = 1$$

$$\int_0^{\infty} x^{3/2}e^{-x}dx = \frac{3\sqrt{\pi}}{4}$$

$$\int_0^{\infty} x^n e^{-x}dx = n!$$