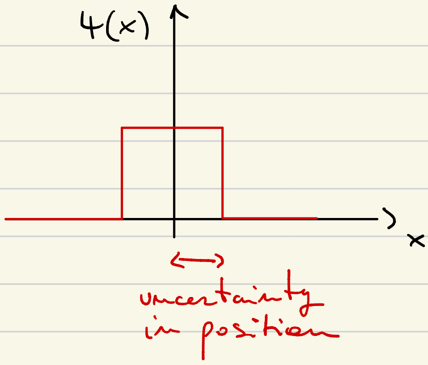
Quantum mechanics I : tutorial solutions

2021 10 07 Self-study pack 1

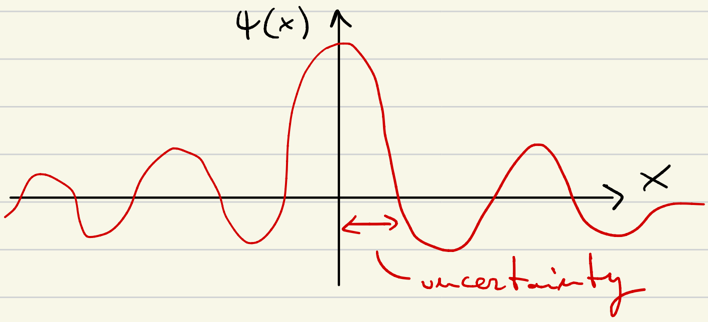
- We want to describe the wave f^{n} of a wave packet.

To do this, we have different possibilities, e.g.

top-hat function

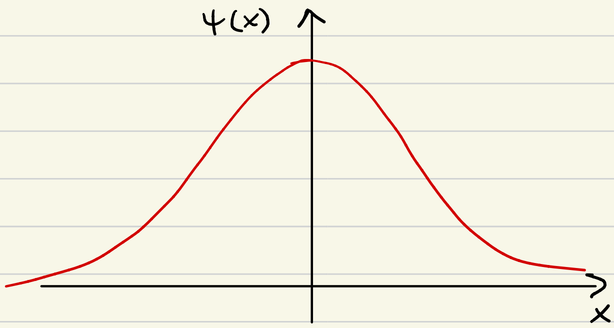


$\frac{\sin(x)}{x}$

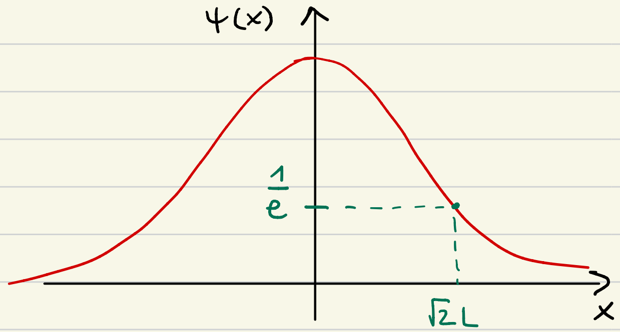


- we look at Gaussian wave packets

$$\psi(x) = \exp\left[-\frac{x^2}{2L^2}\right]$$



1)



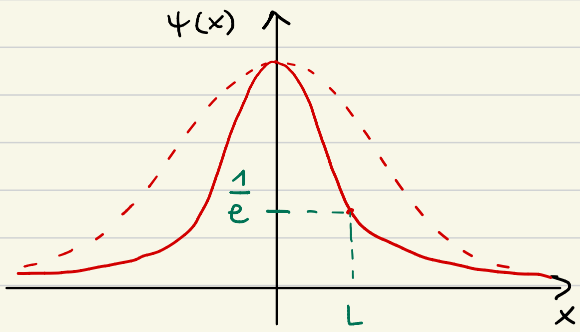
$\exp\left[-\frac{x^2}{2L^2}\right] = \frac{1}{e}$ take the natural logarithm

$-\frac{x^2}{2L^2} = -\underbrace{\ln(e)}_{1}$

$x^2 = 2L^2$ $\quad x = \pm\sqrt{2}\,L$

But what we need to find the probability of finding a particle is $|\psi(x)|^2$ ($\rightarrow$ prob $= \int dx\, |\psi(x)|^2$)

which is $|\psi(x)|^2 = \exp\left[-\frac{x^2}{L^2}\right]$



$\rightarrow$ $\psi(x)$ represents the wave f^n of a particle with position uncertainty L

- Now, we want to write this Gaussian wave function as an INFINITE SUM of plain waves (NO time dependence for now)

$$\psi(x) = \int_{-\infty}^{+\infty} dk \; g(k) \, e^{ikx} \longrightarrow \text{we want to describe } \psi(x) \text{ in terms of plain waves}$$

we want this to be Gaussian

$$= \int dk \; e^{-\frac{\alpha^2 k^2}{2}} e^{ikx} \longrightarrow \text{assume } g(k) \text{ Gaussian}$$

$$= \int dk \; e^{\underbrace{-\frac{\alpha^2 k^2}{2} + ikx}_{\text{COMPLETE THE SQUARE}}}$$

$$-\frac{\alpha^2 k^2}{2} = \left(\frac{i\alpha}{\sqrt{2}} k\right)^2$$

$$ikx = 2\left(\frac{i\alpha}{\sqrt{2}} k\right)\left(\frac{x}{\sqrt{2}\alpha}\right)$$

$$\left(\frac{i\alpha}{\sqrt{2}} k + \frac{x}{\sqrt{2}\alpha}\right)^2 = -\frac{\alpha^2}{2} k^2 + ikx + \frac{x^2}{2\alpha^2}$$

So,

$$\psi(x) = \int_{-\infty}^{+\infty} dk \; \exp\left[\left(\frac{i\alpha k}{\sqrt{2}} + \frac{x}{\sqrt{2}\alpha}\right)^2\right] \exp\left[-\frac{x^2}{2\alpha^2}\right]$$

$$= \exp\left[-\frac{x^2}{2\alpha^2}\right] \int dk \; \exp\left[-\frac{\alpha^2}{2}\left(k - i\frac{x}{\alpha^2}\right)^2\right]$$

change variable $\quad k' = k - i\frac{x}{\alpha^2} \qquad dk' = dk$

$$= \exp\left[-\frac{x^2}{2\alpha^2}\right] \int_{-\infty}^{+\infty} dk' \; \exp\left[-\frac{\alpha^2 k'^2}{2}\right]$$

this is a Gaussian integral. It's solution is

$$\int_{-\infty}^{+\infty} dx \, \exp\left[-a(x+b)^2\right] = \frac{\sqrt{\pi}}{\sqrt{a}}$$

in our case

$$\psi(x) = \exp\left[-\frac{x^2}{2\alpha^2}\right] \int_{-\infty}^{+\infty} dk' \, \exp\left[-\overbrace{\frac{\alpha^2}{2}}^{a} k'^2\right]$$

$$= \frac{\sqrt{2\pi}}{\alpha} \exp\left[-\frac{x^2}{2\alpha^2}\right]$$

Comparing with the f^n we studied before

$$\psi(x) = \exp\left[-\frac{x^2}{2L^2}\right],$$

we see that $\alpha = L$, so the uncertainty in position of the particle is α.

4) the expression of $g(k) = e^{-\frac{\alpha^2 k^2}{2}}$

that with $\alpha = L$

$$g(k) = e^{-\frac{L^2 k^2}{2}}$$

if $\exp\left[-\frac{L^2k^2}{2}\right] = \frac{1}{e}$

take ln: $-\frac{L^2k^2}{2} = -1$

$$\longrightarrow \quad k = \pm\frac{\sqrt{2}}{L}$$

Considering $|g(k)|^2 = e^{-L^2k^2}$

take ln: $-L^2k^2 = -1$

$$\longrightarrow \quad k = \pm\frac{1}{L}$$

UNCERTAINTY IN MOMENTUM k

5) We found the uncertainty in position x,

$$\Delta x = L$$

and in momentum k,

$$\Delta k = \frac{1}{L}$$

the uncertainty relation is $\Delta x \, \Delta k = 1$

Since $p = \hbar k$, this corresponds to

$$\Delta x \Delta p = \hbar$$

- Now we consider the time - dependent version of the wave f^n.

$$\psi(x) = \int_{-\infty}^{+\infty} dk \; e^{-\frac{\alpha^2 k^2}{2}} e^{i(kx - \omega t)}$$

Since we deal with non - relativistic particles ,

$$\hbar \omega = \frac{\hbar^2 k^2}{2m}$$

so that

$$\psi(x) = \int_{-\infty}^{+\infty} dk \; e^{-\frac{\alpha^2 k^2}{2}} e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)}$$

$$= \int dk \; \exp\left[-\frac{1}{2}\left(\alpha^2 + i \frac{\hbar t}{m}\right) k^2 + ikx \right]$$

this is the SAME expression we had in the time - independent case but with

$$\alpha^2 \longrightarrow \alpha^2 + i \frac{\hbar t}{m}$$

Therefore substituting in the final expression

$$\psi(x) = \frac{\sqrt{2\pi}}{\alpha} \exp\left[-\frac{x^2}{2\alpha^2} \right] \qquad (\text{time - independent})$$

$$\psi(x) = \frac{\sqrt{2\pi}}{\sqrt{\alpha^2 + \frac{i\hbar t}{m}}} \exp\left[-\frac{x^2}{2\left(\alpha^2 + i\frac{\hbar t}{m}\right)} \right] \qquad (\text{time dependent})$$

Remove now the "i" from the denominator

$$\psi(x) = \frac{\sqrt{2\pi}}{\sqrt{\alpha^2 + \frac{i\hbar t}{m}}} \sqrt{\frac{\alpha^2 - \frac{i\hbar t}{m}}{\alpha^2 - \frac{i\hbar t}{m}}} \exp\left[-\frac{x^2}{2\left(\alpha^2 + i\frac{\hbar t}{m}\right)} \frac{\alpha^2 - \frac{i\hbar t}{m}}{\alpha^2 - \frac{i\hbar t}{m}} \right]$$

$$= \frac{\sqrt{2\pi}}{\sqrt{\alpha^4 + \frac{\hbar^2 t^2}{m^2}}} \sqrt{\alpha^2 - \frac{i\hbar t}{m}} \exp\left[-\frac{x^2}{2\left(\alpha^4 + \frac{\hbar^2 t^2}{m^2}\right)} \left(\alpha^2 - \frac{i\hbar t}{m}\right) \right]$$

6)

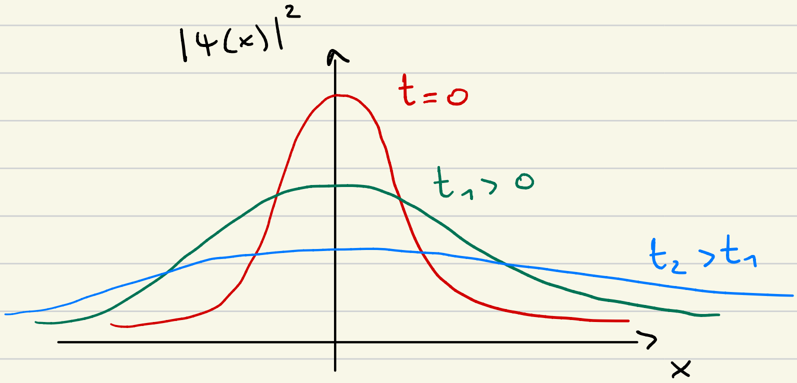
$$|\psi(x)|^2 = \psi^*(x)\, \psi(x)$$

$$= \frac{\sqrt{2\pi}}{\sqrt{\alpha^4 + \frac{\hbar^2 t^2}{m^2}}} \sqrt{\alpha^2 + \frac{i\hbar t}{m}} \exp\left[-\frac{x^2}{2\left(\alpha^4 + \frac{\hbar^2 t^2}{m^2}\right)} \left(\alpha^2 + \frac{i\hbar t}{m}\right) \right]$$

$$\cdot \frac{\sqrt{2\pi}}{\sqrt{\alpha^4 + \frac{\hbar^2 t^2}{m^2}}} \sqrt{\alpha^2 - \frac{i\hbar t}{m}} \exp\left[-\frac{x^2}{2\left(\alpha^4 + \frac{\hbar^2 t^2}{m^2}\right)} \left(\alpha^2 - \frac{i\hbar t}{m}\right) \right]$$

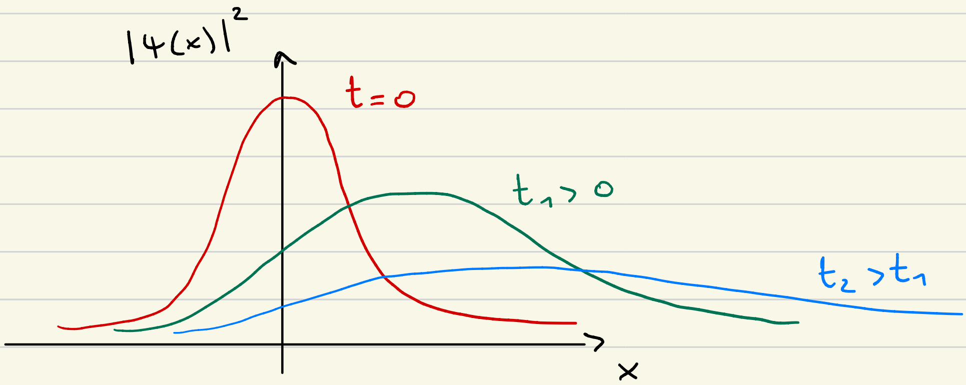
$$= \frac{2\pi}{\sqrt{\alpha^4 + \frac{\hbar^2 t^2}{m^2}}} \exp\left[-\frac{2\alpha^2 x^2}{2\left(\alpha^4 + \frac{\hbar t^2}{m^2}\right)} \right] \exp\left[-\frac{i\hbar t}{m} + \frac{i\hbar t}{m} \right]$$

$$|\psi(x)|^2 = \frac{2\hbar}{\sqrt{\alpha^4 + \frac{\hbar^2 t^2}{m^2}}} \exp\left[- \frac{x^2}{\alpha^2 + \frac{\hbar^2 t^2}{m^2 \alpha^2}} \right]$$



7)

If the gaussian wave packet has a constant momentum, the $|\psi(x)|^2$ shifts while broadening

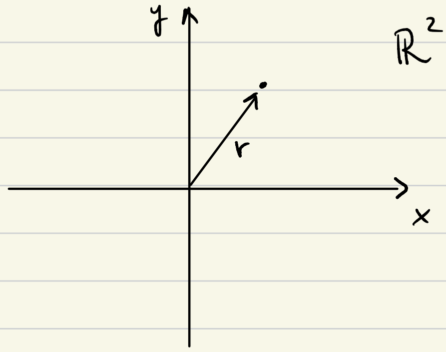


TIPS

- How to do a Gaussian integral $\int_{-\infty}^{+\infty} e^{-x^2}\,dx$

$$\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 = \int e^{-x^2}\,dx \int e^{-y^2}\,dy$$

$$= \int e^{-(x^2+y^2)}\,dx\,dy$$



$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$$= 2\pi \int_0^{\infty} r\,e^{-r^2}\,dr$$

$s = r^2$
$ds = 2r\,dr$

$$= \pi \int_0^{\infty} e^{-s}\,ds$$

$$= \pi\left(e^{-0} - e^{-\infty}\right)$$

$$= \pi$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$