# Quantum mechanics I : tutorial solutions

2021.11.18 / week 8 exercises

1)

1. (a) Write the operators corresponding to the three components $p_x$, $p_y$, and $p_z$ of linear momentum. Use them to derive the operators $\hat{L}_x, \hat{L}_y$ and $\hat{L}_z$ for the components of angular momentum.

(b) State the commutation relations between the operators $\hat{L}_x, \hat{L}_y$ and $\hat{L}_z$ and also between the operator $\hat{L}^2$ and any one of $\hat{L}_x, \hat{L}_y$ and $\hat{L}_z$. Explain the consequences of these relations for the measurements made of the angular momentum of a quantum particle. Compare the results with similar measurements on a classical system.

(c) State the eigenvalue equations for $\hat{L}^2$ and $\hat{L}_z$ in terms of the eigenfunction $Y_{\ell m}(\theta,\phi)$. Calculate the magnitude of the angular momentum of a particle in the state with $\ell = 2$ and $m = 1$.

(a) $$p_i = -i\hbar \partial_i \quad , \quad i = x, y, z$$

in CLASSICAL mechanics, $\underline{L} = \underline{r} \times \underline{p}$

thanks to the CORRESPONDENCE principle of QUANTUM mechanics, we obtain $\hat{\underline{L}}$ by replacing the variables with operators,

$$\hat{\underline{L}} = \hat{\underline{R}} \times \hat{\underline{P}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ -i\hbar\partial_x & -i\hbar\partial_y & -i\hbar\partial_z \end{vmatrix}$$

$$= -i\hbar \left[ \hat{x} \left( y\partial_z - z\partial_y \right) - \hat{y} \left( x\partial_z - z\partial_x \right) + \hat{z} \left( x\partial_y - y\partial_x \right) \right]$$

thus, we have

$$\hat{L}_x = -i\hbar\,(y\,\partial_z - z\,\partial_y)$$
$$\hat{L}_y = -i\hbar\,(z\,\partial_x - x\,\partial_z)$$
$$\hat{L}_z = -i\hbar\,(x\,\partial_y - y\,\partial_x)$$

which is also

$$\hat{L}_x = y\,\hat{P}_z - z\,\hat{P}_y$$
$$\hat{L}_y = z\,\hat{P}_x - x\,\hat{P}_z$$
$$\hat{L}_z = x\,\hat{P}_y - y\,\hat{P}_x$$

(b)

the commutation relations are :

$$[\hat{L}_i, \hat{L}_j] = i\hbar\,\varepsilon_{ijk}\,\hat{L}_k$$

$$[\hat{L}^2, \hat{L}_i] = 0 \quad , \quad \forall\, i$$

These commutation relations tell us that we can measure with arbitrary precision the magnitude of the total angular momentum $\hat{L}^2$ simultaneously with any component $\hat{L}_i$, but we can only have finite precision when measuring, e.g. $\hat{L}_x$ and $\hat{L}_y$

As usual, this is not the case for a classical system for which every precision is allowed.

(c)

the eigenvalue equations are:

$$\hat{L}^2 \, Y_{\ell,m}(\theta, \phi) = \hbar^2 \ell(\ell+1) \, Y_{\ell,m}(\theta,\phi)$$

$$\hat{L}_z \, Y_{\ell,m}(\theta, \phi) = \hbar m \, Y_{\ell,m}(\theta, \phi)$$

where $\hat{L}^2$ & $\hat{L}_z$ in spherical coordinates are

$$\hat{L}^2 = -\hbar^2 \left[ \frac{1}{\sin\theta} \partial_\theta (\sin\theta \, \partial_\theta) + \frac{1}{\sin^2\theta} \partial_\phi^2 \right]$$

$$\hat{L}_z = -i\hbar \, \partial_\phi$$

for a particle with $\ell = 2$, $m = 1$, we have

$$L^2 = \hbar^2 \ell(\ell+1) = 6\hbar^2$$

$$L_z = \hbar m = \hbar$$

2)

2. The operators $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ and $\hat{L}_- = \hat{L}_x - i\hat{L}_y$ are the angular momentum raising and lowering operators respectively (c.f. the raising and lowering operators for the harmonic oscillator). Prove the following commutation relations:

(a) $\left[\hat{L}_z, \hat{L}_+\right] = \hbar \hat{L}_+$

(b) $\left[\hat{L}_z, \hat{L}_-\right] = -\hbar \hat{L}_-$

(c) $\left[\hat{L}^2, \hat{L}_+\right] = \left[\hat{L}^2, \hat{L}_-\right] = 0$

Carry out the following steps to show that the effect of $\hat{L}_+$ is to transform the eigenvalue equation for $\hat{L}_z$ with eigenvalue $m$, into the eigenvalue equation with eigenvalue $m+1$.

(d) Write down the eigenvalue equation for $\hat{L}_z$.

(e) Pre-multiply the equation by $\hat{L}_+$.

(f) Use the commutation relation in (a) above to reverse the order of the operators and hence show that $\left(\hat{L}_z\hat{L}_+ - \hbar\hat{L}_+\right)Y_{\ell m} = m\hbar\hat{L}_+ Y_{\ell m}$.

Rearranging this gives the eigenvalue equation $\hat{L}_z\left(\hat{L}_+ Y_{\ell m}\right) = (m+1)\hbar\left(\hat{L}_+ Y_{\ell m}\right)$ as required.

(a) $$\left[\hat{L}_z, \hat{L}_+\right] = \hat{L}_z\hat{L}_+ - \hat{L}_+\hat{L}_z$$

here, we expand the commutator

$$= \hat{L}_z(\hat{L}_x + i\hat{L}_y) - (\hat{L}_x + i\hat{L}_y)\hat{L}_z$$
$$= \hat{L}_z\hat{L}_x - \hat{L}_x\hat{L}_z + i(\hat{L}_z\hat{L}_y - \hat{L}_y\hat{L}_z)$$
$$= [\hat{L}_z, \hat{L}_x] - i[\hat{L}_y, \hat{L}_z]$$
$$= i\hbar\hat{L}_y + \hbar\hat{L}_x$$
$$= \hbar\hat{L}_+$$

(b) $$\left[\hat{L}_z, \hat{L}_-\right] = \left[\hat{L}_z, \hat{L}_x - i\hat{L}_y\right]$$

here, we keep the the commutator implicit

$$= [\hat{L}_z, \hat{L}_x] - i[\hat{L}_z, \hat{L}_y]$$
$$= i\hbar\hat{L}_y - i(i\hbar)(-\hat{L}_x)$$
$$= -\hbar(\hat{L}_x - i\hat{L}_y)$$
$$= -\hbar\hat{L}_-$$

(c) $[\hat{L}^2, \hat{L}_\pm] = [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 , \hat{L}_x \pm i \hat{L}_y]$

$$= [\hat{L}_x^2, \hat{L}_x] \pm i [\hat{L}_x^2, \hat{L}_y]$$
$$+ [\hat{L}_y^2, \hat{L}_x] \pm i [\hat{L}_y^2, \hat{L}_y]$$
$$+ [\hat{L}_z^2, \hat{L}_x] \pm i [\hat{L}_z^2, \hat{L}_y]$$

$$= 0 \pm i \left( \hat{L}_x [\hat{L}_x, \hat{L}_y] + [\hat{L}_x, \hat{L}_y] \hat{L}_x \right)$$
$$+ \hat{L}_y [\hat{L}_y, \hat{L}_x] + [\hat{L}_y, \hat{L}_x] \hat{L}_y \pm 0$$
$$+ \hat{L}_z [\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_x] \hat{L}_z$$
$$\pm i \left( \hat{L}_z [\hat{L}_z, \hat{L}_y] + [\hat{L}_z, \hat{L}_y] \hat{L}_z \right)$$

$$= \pm i (i\hbar) (\hat{L}_x \hat{L}_z + \hat{L}_z \hat{L}_x)$$
$$- i\hbar (\hat{L}_y \hat{L}_z + \hat{L}_z \hat{L}_y)$$
$$+ i\hbar (\hat{L}_z \hat{L}_y + \hat{L}_y \hat{L}_z)$$
$$\pm i (-i\hbar) (\hat{L}_z \hat{L}_x + \hat{L}_x \hat{L}_z)$$

$$= 0$$

(d) eigenvalue eq. for $\hat{L}_z$

$$\hat{L}_z \, \psi(\ell, m) = \hbar m \, \psi(\ell, m)$$

(e) premultiply by $\hat{L}_+$

$$\hat{L}_+ \hat{L}_z \, \psi(\ell, m) = \hbar m \, \hat{L}_+ \psi(\ell, m)$$

using the commutation relation we found in (a)

$$[\hat{L}_z, \hat{L}_+] = \hat{L}_z \hat{L}_+ - \hat{L}_+ \hat{L}_z = \hbar \hat{L}_+$$

$$\hat{L}_+ \hat{L}_z = \hat{L}_z \hat{L}_+ - \hbar \hat{L}_+$$

we have

$$\hat{L}_z \hat{L}_+ \psi(\ell, m) - \hbar \hat{L}_+ \psi(\ell, m) = \hbar m \, \hat{L}_+ \psi(\ell, m)$$

$$\hat{L}_z \, \hat{L}_+ \psi(\ell, m) = \hbar (m + 1) \, \hat{L}_+ \psi(\ell, m)$$

3)

3. The classical expression for the kinetic energy of a rigid rotating body is

$$T = \frac{1}{2}\left(\frac{L_x^2}{I_x} + \frac{L_y^2}{I_y} + \frac{L_z^2}{I_z}\right)$$

where $L_x$, $L_y$ and $L_z$ are the components of angular momentum and $I_x$, $I_y$ and $I_z$ are the principal moments of inertia.

(a) Set up the corresponding Hamiltonian operator for the quantum mechanical situation.

(b) Determine the eigenvalues of this operator for the case in which the body is spherical, i.e. $I_x = I_y = I_z$. Give an expression for the degeneracy of the eigenstate with total angular momentum quantum number $\ell$.

(c) Find an expression for the eigenvalues of the Hamiltonian operator if the body has $I_x = I_y = 2I_z$ and calculate the eigenvalues and degeneracies for the states with $\ell = 2$.

(a) for the qm version, we substitute the variables of the system w/ operators,

$$\hat{H} = \frac{1}{2}\left(\frac{\hat{L}_x^2}{I_x} + \frac{\hat{L}_y^2}{I_y} + \frac{\hat{L}_z^2}{I_z}\right)$$

(b) If we consider the spherically symmetric case, we have

$$\hat{H} = \frac{1}{2I}\left(\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2\right) = \frac{\hat{L}^2}{2I}$$

the eigenvalue equation for the operator above for a particle with the quantum number of total angular momentum $\ell$ is

$$\hat{H}\psi(\ell, m) = \frac{\hat{L}^2}{2I}\psi(\ell, m) = \frac{\hbar^2 \ell(\ell+1)}{2I}\psi(\ell, m)$$

Since

$$m \in \underbrace{\{-l, -l+1, \dots, 0, \dots, l-1, l\}}_{2l+1}$$

the degeneracy is $2l+1$

(c) If the body has $I_x = I_y = 2I_z$

the $\hat{H}$ operator is

$$\hat{H} = \frac{1}{2}\left(\frac{\hat{L}_x^2}{2I_z} + \frac{\hat{L}_y^2}{2I_z} + \frac{\hat{L}_z^2}{I_z}\right)$$

$$= \frac{1}{4I_z}\left(\hat{L}_x^2 + \hat{L}_y^2 + 2\hat{L}_z^2\right)$$

$$= \frac{1}{4I_z}\left(\hat{L}^2 + \hat{L}_z^2\right)$$

the corresponding eigenvalue equation is

$$\hat{H}\,\psi(l,m) = \frac{1}{4I_z}\left(\hat{L}^2 + \hat{L}_z^2\right)\psi(l,m)$$

$$= \frac{1}{4I_z}\left(\hbar^2 l(l+1) + \hbar^2 m^2\right)\psi(l,m)$$

$$= \frac{\hbar^2\left(l(l+1) + m^2\right)}{4I_z}\,\psi(l,m)$$

for a particle w/ $\ell = 2$, we have

degeneracy $2\ell + 1 = 5$

the possible states are

$(\ell = 2, m = 0)$ $E_{2,0} = \frac{3}{2} \frac{\hbar^2}{I_z}$, deg $= 1$

$(\ell = 2, m = \pm 1)$ $E_{2,\pm 1} = \frac{7}{4} \frac{\hbar^2}{I_z}$, deg $= 2$

$(\ell = 2, m = \pm 2)$ $E_{2,\pm 2} = \frac{5}{2} \frac{\hbar^2}{I_z}$, deg $= 2$

tips

## COMMUTATOR PROPERTIES

$[a, b] = ab - ba$ when in doubt, use the definition

$[a, a] = 0$

$[a, b] = -[b, a]$

$[a+b, c] = [a, c] + [b, c]$

$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ Jacobi identity

$[\alpha a, b] = \alpha [a, b]$ where $\alpha$ is a constant

$[ab, c] = a[b, c] + [a, c]b$ watch out where they go out