PHY2022

UNIVERSITY OF EXETER

PHYSICS

JANUARY 2017

QUANTUM MECHANICS I

Duration: TWO HOURS

Answer ALL four questions. Full marks (100) are attained with four complete answers. (Marks may be subject to scaling by the APAC.)

Use a single answer book for all questions (1 book).

Materials to be supplied:

Physical Constants sheet

Approved calculators are permitted

This is a ‘closed note’ examination

1. Discuss the physical interpretation of the wavefunction in quantum mechanics. **[3]**

Describe the procedure by which a wavefunction is normalized, and explain the physical reason for performing this procedure. **[3]**

A wave packet is described by the wavefunction

$$\psi(x) = \begin{cases} B\left(a^2 - x^2\right) & \text{for } -a \le x \le a \\ 0 & \text{otherwise,} \end{cases}$$

where a and B are constants.

For this wave packet:

(a) Show that a value of B that normalizes the wavefunction is $\frac{\sqrt{15}}{4}a^{-5/2}$. **[5]**

(b) Calculate the expectation value $\left\langle x^2 \right\rangle$. **[5]**

(c) Calculate the expectation value of the square of the momentum, $\left\langle p_x{}^2 \right\rangle$. **[5]**

(d) Given that the expectation values of x and p_x are both zero, and that the standard deviation in a quantity q is defined as $\Delta q = \sqrt{\left\langle q^2 \right\rangle - \left\langle q \right\rangle^2}$, evaluate the uncertainty product $\Delta x \Delta p_x$ and comment on how this result relates to the uncertainty principle. **[4]**

2. A particle of mass m is confined within an infinite square-well potential of width $2a$ centred about $x = 0$, having zero potential energy within the well.

(a) Calculate the normalized eigenfunctions, $u_n(x)$ (distinguishing the cases where n is even and odd), and the eigenvalues, E_n, of the stationary states of the particle. **[8]**

(b) Sketch the wavefunctions, and the probability densities for a position measurement, for the first excited state $u_2(x)$ and the third excited state $u_4(x)$. **[4]**

The particle in this well is set up in the superposition state

$$\psi(x) = A(2u_2 - 3i\,u_4).$$

(c) Calculate a value for A that normalizes $\psi(x)$. **[3]**

(d) What are the possible outcomes of a measurement of energy on this state, and what are the probabilities of each outcome? **[4]**

What is the probability density for a position measurement of the particle in the state $\psi(x)$:

(e) at $x = 0$; **[2]**

(f) at $x = a/2$. **[4]**

3. Consider a particle of mass m confined to move in one dimension and subjected to a harmonic oscillator potential $V(x) = \frac{1}{2}kx^2$.

(a) Write down the time-independent Schrödinger equation for this system. **[3]**

(b) Find the relationships between ω and k, and between ω and the total energy E, for which

$$u(x) = A\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

is a solution of the Schrödinger equation from (a). **[5]**

(c) Sketch the dependence of $u(x)$ on x. What aspect of your sketch indicates that $u(x)$ is the ground state? **[3,1]**

(d) Write down the energy of the nth excited state for the particle subjected to this potential. **[2]**

(e) Calculate a value of A that normalizes $u(x)$. **[4]**

[You may use the following standard integral: $\int_{-\infty}^{\infty}\exp(-ax^2)\,dx = (\pi/a)^{1/2}$]

Another particle, also of mass m, is confined to move in two dimensions and subjected to the potential $V(x,y) = \frac{1}{2}(k_1x^2 + k_2y^2)$.

(f) Write down an expression for the energy of the state having quantum numbers n_1 and n_2 associated with motion in the x and y directions respectively. **[2]**

(g) Determine the energies and degeneracies of the three lowest energy levels of this system for the special case $k_1 = k_2$. **[5]**

Turn Over

4. One of the quantum numbers associated with the hydrogen atom is the principal quantum number, n.

(a) Name the other two quantum numbers and state the restrictions on the values of all three. **[3]**

(b) How many distinct hydrogen atom eigenstates are there having principal quantum number 4? **[3]**

The normalized wavefunction for the ground state of the electron in a hydrogen atom is

$$u_{100}(r,\theta,\phi) = \frac{1}{\left(\pi a_0{}^3\right)^{1/2}} \exp\left(-\frac{r}{a_0}\right),$$

where a_0 is the Bohr radius.

(c) What is the probability that a measurement of the distance between the electron and the proton in this state yields a value between r and $r + dr$? **[3]**

(d) Calculate the probability of finding an electron in the state described by u_{100} within one Bohr radius of the proton. **[8]**

Protonium is a system consisting of a proton of mass m_{P} bound by the strong nuclear force to an anti-proton, also of mass m_{P}. The proton–anti-proton potential is of the form

$$V(r) = -\frac{a}{r}\exp(-br),$$

where r is the distance between the proton and the anti-proton.

(e) Given that the Bohr radius and energy levels of the hydrogen atom are

$$a_0 = \left(\frac{4\pi\varepsilon_0}{e^2}\right)\frac{\hbar^2}{m} \text{ and } E_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{1}{n^2},$$

where m is the reduced mass of the electron–proton system, derive expressions for the Bohr radius and energy levels of protonium, in the approximation that $b = 0$. **[8]**

END OF PAPER