1. Discuss the physical interpretation of the wavefunction in quantum mechanics. **[3]**

Describe the procedure by which a wavefunction is normalized, and explain the physical reason for performing this procedure. **[3]**

A wave packet is described by the wavefunction

$$\psi(x) = \begin{cases} B\left(a^2 - x^2\right) & \text{for } -a \le x \le a \\ 0 & \text{otherwise,} \end{cases}$$

where a and B are constants.

For this wave packet:

(a) Show that a value of B that normalizes the wavefunction is $\frac{\sqrt{15}}{4}a^{-5/2}$. **[5]**

(b) Calculate the expectation value $\langle x^2 \rangle$. **[5]**

(c) Calculate the expectation value of the square of the momentum, $\langle p_x{}^2 \rangle$. **[5]**

(d) Given that the expectation values of x and p_x are both zero, and that the standard deviation in a quantity q is defined as $\Delta q = \sqrt{\langle q^2 \rangle - \langle q \rangle^2}$, evaluate the uncertainty product $\Delta x \Delta p_x$ and comment on how this result relates to the uncertainty principle. **[4]**

- ψ is a probability amplitude
- $|\psi|^2$ is a probability density
- $|\psi(x)|^2\, dx$ probability of finding the particle in $[x, x+dx]$

normalization condition : $\displaystyle\int_{\text{all space}} |\psi(x)|^2\, dx = 1$

The reason we do this is that we want the probability of finding the particle in its domain to be $\underline{\underline{1}}$

$$\psi(x) = \begin{cases} B\,(a^2 - x^2) & \text{for} \quad -a \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

(a) normalization:

$$B^2 \int_{-a}^{a} (a^2 - x^2)^2 \, dx = 1$$

$$= B^2 \left[a^4 \int dx + \int x^4 \, dx - 2a^2 \int x^2 \, dx \right]$$

$$= B^2 \left[2a^5 + \frac{2}{5} a^5 - \frac{4}{3} a^5 \right]$$

$$= B^2 \left[\frac{30 + 6 - 20}{15} a^5 \right]$$

$$= B^2 \, \frac{16}{15} \, a^5 = 1$$

$$\rightarrow \quad B = \frac{\sqrt{15}}{4} a^{-5/2}$$

(b) $$\langle x^2 \rangle = \int_{-a}^{a} B^2 (a^2 - x^2) \, x^2 (a^2 - x^2) \, dx$$

$$= B^2 \int (a^2 - x^2)(a^2 x^2 - x^4) \, dx$$

$$= B^2 \int (a^4 x^2 - a^2 x^4 - a^2 x^4 + x^6) \, dx$$

$$= B^2 a^7 \left(\frac{2}{3} - \frac{4}{5} + \frac{2}{7} \right)$$

$$= \frac{15}{16} a^{-5} \, a^7 \, \frac{16}{105} = \frac{1}{7} a^2$$

(c) $\langle p_x^2 \rangle = B^2 \int (a^2 - x^2)(-\hbar^2 \partial_x^2)(a^2 - x^2)\, dx$

$= -\hbar^2 B^2 \int (a^2 - x^2)(-2)\, dx$

$= 2\hbar^2 B^2 \left[\int a^2\, dx - \int x^2\, dx \right]$

$= 2\hbar^2 B^2 \left[2a^3 - \frac{2}{3} a^3 \right]$

$= \frac{8}{3} \hbar^2 B^2 a^3$

$= \frac{8}{3} \hbar^2 \frac{15}{16} a^{-5} a^3 = \frac{15}{6} \hbar^2 a^{-2}$

(d) $\langle p_x \rangle = \int_{-a}^{a} B^4 (a^2 - x^2) -i\hbar \partial_x (a^2 - x^2)\, dx$

$= -i\hbar B^4 \int \underbrace{(a^2 - x^2)(-2x)}_{\text{odd functions}}\, dx$

$\int_{-a}^{a} \text{odd} = 0$

$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{1}{7} a^2}$

$\Delta p_x = \sqrt{\langle p_x^2 \rangle - \langle p_x \rangle^2} = \sqrt{\frac{15}{6} \hbar^2 a^{-2}}$

$\Delta x \Delta p_x = \sqrt{\frac{15}{42}}\, \hbar$

3. Consider a particle of mass m confined to move in one dimension and subjected to a harmonic oscillator potential $V(x) = \frac{1}{2}kx^2$.

(a) Write down the time-independent Schrödinger equation for this system. **[3]**

(b) Find the relationships between ω and k, and between ω and the total energy E, for which

$$u(x) = A\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

is a solution of the Schrödinger equation from (a). **[5]**

(c) Sketch the dependence of $u(x)$ on x. What aspect of your sketch indicates that $u(x)$ is the ground state? **[3,1]**

(d) Write down the energy of the nth excited state for the particle subjected to this potential. **[2]**

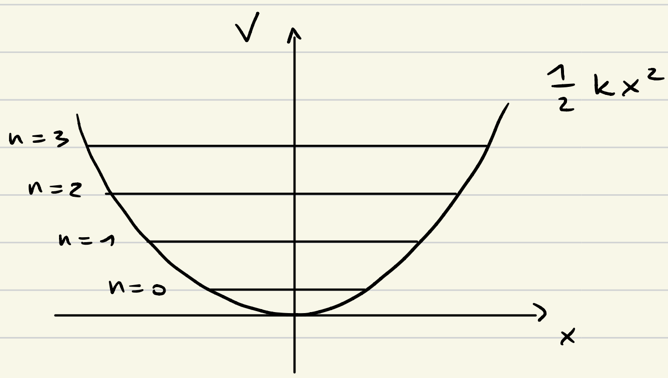
(e) Calculate a value of A that normalizes $u(x)$. **[4]**

[You may use the following standard integral: $\int_{-\infty}^{\infty} \exp(-ax^2)\,dx = (\pi/a)^{1/2}$]

Another particle, also of mass m, is confined to move in two dimensions and subjected to the potential $V(x,y) = \frac{1}{2}(k_1x^2 + k_2y^2)$.

(f) Write down an expression for the energy of the state having quantum numbers n_1 and n_2 associated with motion in the x and y directions respectively. **[2]**

(g) Determine the energies and degeneracies of the three lowest energy levels of this system for the special case $k_1 = k_2$. **[5]**



$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$

(a) time-independent Schrödinger eq.

$\hat{H}\psi = E\psi$

$$\hat{H} = \frac{p_x^2}{2m} + \frac{1}{2} k x^2 \qquad \omega = \sqrt{\frac{k}{m}}$$

$$= -\frac{\hbar^2}{2m} \partial_x^2 + \frac{1}{2} m \omega^2 x^2$$

$$\hat{H} \psi(x) = E \psi(x)$$

$$-\frac{\hbar^2}{2m} \partial_x^2 \psi(x) + \frac{1}{2} m \omega^2 x^2 \psi(x) = E \psi(x)$$

(b) $$u(x) = A e^{-\frac{m\omega x^2}{2\hbar}}$$

$$-\frac{\hbar^2}{2m} \partial_x^2 u(x) + \frac{1}{2} k x^2 u(x) = E u(x)$$

$$-\frac{\hbar^2}{\cancel{2} m} \left(-\frac{m\omega}{2\hbar}\right) \partial_x \left(\cancel{2} x \; u(x)\right) + \frac{1}{2} k x^2 u(x) = E u(x)$$

$$\boxed{-\frac{\hbar^2}{m} \frac{m^2 \omega^2}{4\hbar^2} 2 x^2 u(x) + \frac{\hbar^2}{m} \frac{m\omega}{2\hbar} u(x) + \frac{1}{2} k x^2 u(x) = E u(x)}$$

To be an eigenfunction, the pieces dependent on x^2 need to cancel out.

$$\frac{1}{2} k \cancel{x^2} - \frac{m\omega^2}{2} \cancel{x^2} = 0$$

$$k = m\omega^2 \longrightarrow \omega = \sqrt{\frac{k}{m}}$$

On the other end, we are left with

$$E\, u(x) = \frac{\hbar^2}{m} \frac{m\omega}{2\hbar} u(x)$$

therefore

$$E = \frac{\hbar\omega}{2}$$

which is the energy of $u(x)$, i.e. the ground state of the h.o.

To find $E(\omega)$, We write $\hat{x}$ & $\hat{p}$ as:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a}^{+} + a \right)$$

$$\hat{p} = i \sqrt{\frac{\hbar m\omega}{2}} \left(\hat{a}^{+} - \hat{a} \right)$$

which gives,

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right)$$

$$\hat{a}^{+} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right)$$

rewriting the Hamiltonian in the new variables,

$$\hat{H} = \hbar\omega \left(\underbrace{\hat{a}^{+} \hat{a}}_{N} + \frac{1}{2} \right)$$

where $\hat{a}^{+} |n\rangle = \sqrt{n+1} \; |n+1\rangle$

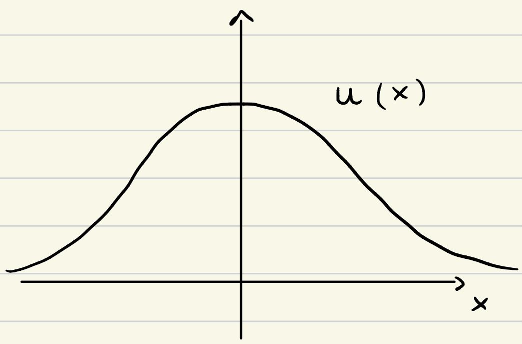
$\hat{a} \; |n\rangle = \sqrt{n} \; |n-1\rangle$

the new $\hat{H}$ tells me that

$$\hat{H} \, |n\rangle = \hbar\omega \left(N + \frac{1}{2} \right) |n\rangle = E \, |n\rangle$$

$$\longrightarrow \quad E = \hbar\omega \left(n + \frac{1}{2} \right)$$

(c)



$u(x)$ is a gaussian wave function, meaning it is multiplied by the first Hermite polynomial $H_0(x) = 1$ (while $H_1(x) = 2x$, $H_2(x) = 4x^2 - 1$, etc.)

To find the ground state u_0 we need to find the solution to $\hat{a}\, u_0 = 0$

In x, p coordinates,

$$\hat{a}\, u_0 = \frac{m\omega}{2\hbar}\left(x + \frac{i}{m\omega}\,\hat{p}\right) u_0$$

$$= \frac{m\omega}{2\hbar}\left(x + \frac{\hbar}{m\omega}\,\partial_x\right) u_0$$

$$= \left(\frac{m\omega}{2\hbar}\, x + \frac{1}{2}\,\partial_x\right) u_0 = 0$$

$$\rightarrow \quad \partial_x u_0 = \frac{m\omega}{\hbar}\, u_0$$

$$\text{sol:} \quad u_0 = A\, e^{-\frac{m\omega}{2\hbar} x^2}$$

(d) the n-th excited state is the Hermite f^n

$$u_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}\, H_n\!\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right)$$

with $n = 0, 1, 2, \ldots$

Here, H_n is a Hermite polynomial of degree n

(e) Normalization:

$$\int_{-\infty}^{+\infty} A^2 e^{-\frac{m\omega x^2}{\hbar}}\, dx = 1$$

$$\int e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \longrightarrow \int e^{-\frac{m\omega x^2}{\hbar}}\, dx = \sqrt{\frac{\pi\hbar}{m\omega}}$$

$$A^2 \sqrt{\frac{\pi \hbar}{m \omega}} = 1$$

$$A = \left(\frac{m \omega}{\hbar \pi} \right)^{1/4}$$

(f) consider $V(x, y) = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2$

we know that, if the potential is separable, the total energy is the sum of the energies,

$$E_{n_1, n_2} = \hbar \omega_1 \left(n_1 + \frac{1}{2} \right) + \hbar \omega_2 \left(n_2 + \frac{1}{2} \right)$$

(g) In the special case $k_1 = k_2$

$$E_{n_1, n_2} = \hbar \omega \left(n_1 + n_2 + 1 \right)$$

$E_{0,0} = \hbar \omega$, deg $= 1$

$E_{1,0} = E_{0,1} = 2 \hbar \omega$, deg $= 2$

$E_{2,0} = E_{0,2} = E_{1,1} = 3 \hbar \omega$, deg $= 3$