Thermal Physics : tutorial solution

2020.02.26

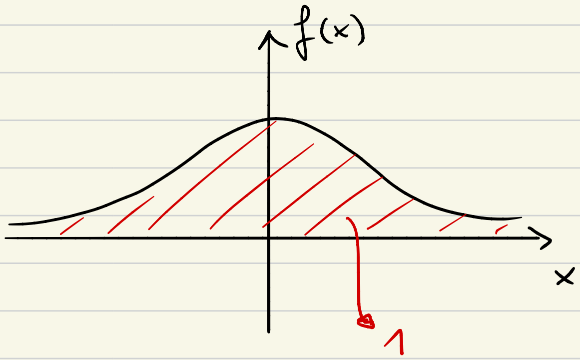
2020.03.04

1 a) Show that the velocity distribution f^{n} is NORMALISED.

$$g(v_x) = \sqrt{\frac{m}{2\pi k_B T}} \exp\left[-\frac{m v_x^2}{2 k_B T}\right]$$

NB condition of normalisation:

$$\int_D dx \; f(x) = 1$$



where D is the domain of $f(x)$

$$\int_{-\infty}^{+\infty} dv_x \; g(v_x) = \int dv_x \sqrt{\frac{m}{2\pi k_B T}} \exp\left[-\underbrace{\frac{m v_x^2}{2 k_B T}}_{\alpha^{-1}}\right]$$

$$= \sqrt{\frac{1}{2\pi\alpha}} \underbrace{\int dv_x \exp\left[-\frac{v_x^2}{2\alpha}\right]}_{\text{check table of integrals}}$$

where $\alpha = \frac{k_B T}{m} \geq 0$

Using $\int_{-\infty}^{+\infty} dx \exp\left[-\frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2}$,

$$\int_{-\infty}^{+\infty} dv_x \; g(v_x) = \sqrt{\frac{1}{2\pi\alpha}} \sqrt{2\pi\alpha} = 1 \longrightarrow \text{NORMALISED}$$

1b) Show that $\langle v_x \rangle = 0$

NB $\langle v_x \rangle$ is called 1^{st} moment of v_x

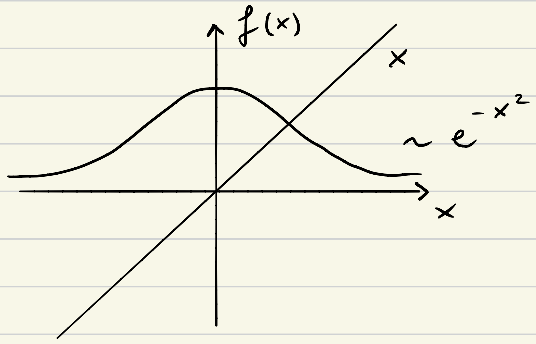
It is defined as $\langle v_x \rangle = \int_{-\infty}^{+\infty} dv_x \; v_x \; g(v_x)$

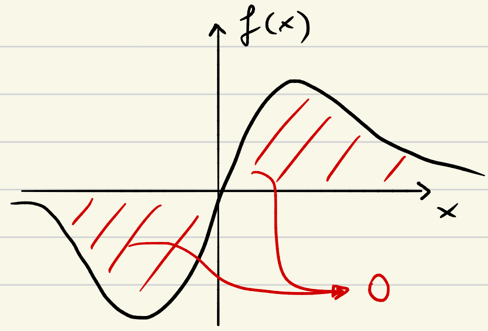
$$\int_{-\infty}^{+\infty} dv_x \; v_x \; g(v_x) = \int dv_x \sqrt{\frac{m}{2\pi k_B T}} \underbrace{\underbrace{v_x}_{\text{odd}} \; \underbrace{\exp\left[-\frac{m v_x^2}{2 k_B T}\right]}_{\text{even}}}_{\text{odd}}$$

$$= 0$$

OK

The integrand is an ODD f^n on the domain of definition. The integral on the entire domain is therefore 0.





Show that $\langle v_x^2 \rangle = \frac{k_B T}{m}$

NB $\langle v_x^2 \rangle$ is called 2^{nd} moment of v_x

It is defined as $\langle v_x^2 \rangle = \int_{-\infty}^{+\infty} dv_x \; v_x^2 \; g(v_x)$

$$\int_{-\infty}^{+\infty} dv_x \; v_x^2 \; g(v_x) = \int dv_x \sqrt{\frac{m}{2\pi k_B T}} \; v_x^2 \; \exp\left[-\frac{m v_x^2}{2 k_B T}\right]$$

α^{-1}

$$= \sqrt{\frac{1}{2\pi\alpha}} \int dv_x \; v_x^2 \; \exp\left[-\frac{v_x^2}{2\alpha}\right]$$

We want to use the integral:

$$\int_0^{\infty} dx \; x^2 \; e^{-x^2} = \frac{\sqrt{\pi}}{4} ,$$

but before, we have to change variable and "fix" the domain of integration.

$$y^2 = \frac{v_x^2}{2\alpha} , \qquad v_x^2 = 2\alpha y^2 , \qquad v_x = \sqrt{2\alpha}\, y$$

$$dv_x = \sqrt{2\alpha}\, dy$$

the integral becomes

$$\int_{-\infty}^{+\infty} dv_x \; v_x^2 \; g(v_x) = \frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{+\infty} \sqrt{2\alpha}\, dy \; 2\alpha y^2 \; \exp[-y^2]$$

$$= \frac{2\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} dy \; \underbrace{y^2 \exp[-y^2]}_{\text{even}}$$

NB Since the f^{n} is EVEN,

$$\int_{-\infty}^{+\infty} dy\; y^2\, e^{-y^2} = 2 \int_{0}^{\infty} dy\; y^2\, e^{-y^2}$$

$$\Rightarrow \int_{-\infty}^{+\infty} dv_x\; v_x^2\, g(v_x) = \frac{2\alpha}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dy\; y^2\, e^{-y^2}$$

$$= \frac{4\alpha}{\sqrt{\pi}} \underbrace{\int_{0}^{\infty} dy\; y^2\, e^{-y^2}}_{\sqrt{\frac{\pi}{4}}}$$

$$= \frac{4\alpha}{\sqrt{\pi}} \frac{\sqrt{\pi}}{4}$$

$$= \alpha$$

$$= \frac{k_B T}{m}$$ OK

2) the Maxwell - Boltzmann distribution for an ideal gas is:

$$p(\varepsilon)\, d\varepsilon = \frac{2}{\sqrt{\pi}} \beta^{3/2} \sqrt{\varepsilon} \exp[-\beta\varepsilon]\, d\varepsilon$$

2a) Normalisation condition: $\int_{\text{Domain}} f(x)\, dx = 1$

2b) Show that $p(\varepsilon)$ is properly normalised.

$$\int_0^{+\infty} d\varepsilon\, p(\varepsilon) = \frac{2}{\sqrt{\pi}} \beta^{3/2} \int_0^{+\infty} d\varepsilon \sqrt{\varepsilon} \exp[-\beta\varepsilon]$$

we want to use the integral $\int_0^{+\infty} \sqrt{x}\, e^{-x}\, dx = \frac{\sqrt{\pi}}{2}$

but before we have to change variable.

$$y = \beta\varepsilon \quad , \quad \sqrt{\varepsilon} = \frac{1}{\sqrt{\beta}} \sqrt{y} \quad , \quad dy = \beta\, d\varepsilon \quad , \quad d\varepsilon = \frac{1}{\beta} dy$$

$$\int_0^{+\infty} d\varepsilon\, p(\varepsilon) = \frac{2}{\sqrt{\pi}} \beta^{3/2} \int_0^{+\infty} \frac{1}{\beta} dy \frac{1}{\sqrt{\beta}} \sqrt{y} \exp[-y]$$

$$= \frac{2}{\sqrt{\pi}} \underbrace{\int_0^{+\infty} dy \sqrt{y}\, e^{-y}}_{\frac{\sqrt{\pi}}{2}}$$

$$= 1$$

OK

2c) Using $\int_0^{+\infty} x^{3/2} e^{-x} dx = \frac{3\sqrt{\pi}}{4}$

& equipartition theorem, show that $\beta = \frac{1}{k_B T}$

The equipartition theorem relates the average energy per particle with the temperature of the gas in thermal equilibrium.

For a MONO ATOMIC gas in 3 DIMENSIONS is:

$$\langle \varepsilon \rangle = \frac{3}{2} k_B T$$

in general, $\langle \varepsilon \rangle = \underbrace{dof}_{} \cdot \frac{1}{2} k_B T$

every quadratic degree of freedom (in this case 3)

We calculate $\langle \varepsilon \rangle$, the 1st moment of ε

$$\langle \varepsilon \rangle = \int_0^{+\infty} d\varepsilon \; \varepsilon \, p(\varepsilon) = \frac{2}{\sqrt{\pi}} \beta^{3/2} \int_0 d\varepsilon \; \varepsilon^{3/2} \exp[-\beta \varepsilon] \quad ,$$

we change variable

$$y = \beta \varepsilon \quad , \quad \varepsilon = \frac{1}{\beta} y \quad , \quad d\varepsilon = \frac{1}{\beta} dy$$

$$\int_0^{+\infty} d\varepsilon \; \varepsilon \, p(\varepsilon) = \frac{2}{\sqrt{\pi}} \beta^{3/2} \int_0^{+\infty} \frac{1}{\beta} dy \, \frac{1}{\beta^{3/2}} y^{3/2} \quad e^{-y}$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{\beta} \underbrace{\int_0^{+\infty} dy \; y^{3/2} \; e^{-y}}_{\frac{3}{4}\sqrt{\pi}}$$

$$\langle \varepsilon \rangle = \frac{3}{2} \frac{1}{\beta}$$

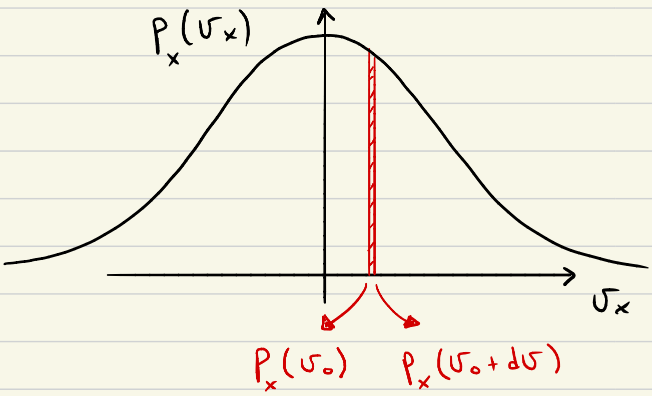
and from equipartition theorem $\langle \varepsilon \rangle = \frac{3}{2} k_B T$

$$\frac{3}{2} \frac{1}{\beta} = \frac{3}{2} k_B T \quad \Rightarrow \quad \beta = \frac{1}{k_B T}$$

OK

3a) State the meaning of $p_x(v_x)\,dv_x$.

$p_x(v_x)\,dv_x$ is the probability of finding a particle with velocity between $[v_x, v_x + dv_x]$.



3b) Assume that $p_x(v_x) = \exp[-a v_x^2]$

Show that for IDEAL 2D gas the speed distr. is:

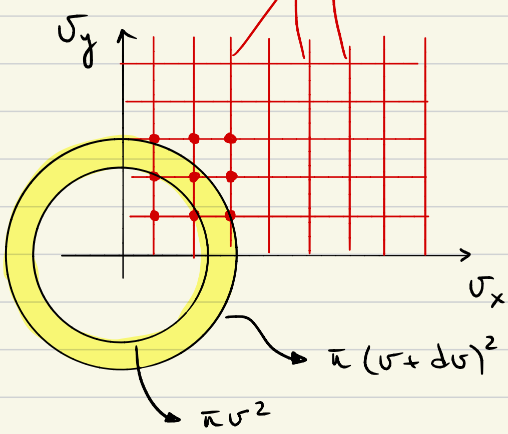
$$p(v)\,dv = f\,v\,\exp[-a v^2]\,dv$$

with $v = \sqrt{v_x^2 + v_y^2}$ and f is CONSTANT.

- The 2-D probability of finding a particle with velocity between $[v_x, v_x + dv_x]$ & $[v_y, v_y + dv_y]$ is:

$$\underbrace{p(v_x)\,p(v_y)}_{p(v)}\;\underbrace{dv_x\,dv_y}_{\propto v\,dv\,d\phi}$$

In more detail, in VELOCITY - space we have:



Considering this states infinitely close each other, we can consider a continuous space.

We want to find the state in an infinitesimal portion of area in the velocity space. We do this in polar coordinates.

The highlighted infinitesimal area is:

$$\underbrace{\pi(v+dv)^2}_{\text{LARGE area}} - \underbrace{\pi v^2}_{\text{SMALL area}} = \pi(\cancel{v^2} + \underbrace{\cancel{dv^2}}_{2^{nd}\text{ order}} + 2v\,dv - \cancel{v^2})$$

$$= 2\pi v\,dv$$

from here, we have the "v" dependence

3c) Find f by requiring $p(v)$ NORMALISED

$$f \int_0^{+\infty} dv \; v \exp[-a v^2]$$

NB { the integral is $\int_0^{+\infty}$ and not $\int_{-\infty}^{+\infty}$ because $\underbrace{v}_{>0} = \sqrt{v_x^2 + v_y^2}$

changing $y = a v^2$, $v = \frac{1}{\sqrt{a}} \sqrt{y}$, $dv = \frac{1}{2\sqrt{a}} \frac{1}{\sqrt{y}} dy$

$$= f \int_0^{+\infty} dy \; \frac{1}{2\sqrt{a}} \frac{1}{\sqrt{y}} \frac{1}{\sqrt{a}} \sqrt{y} \; e^{-y}$$

$$= f \frac{1}{2a} \int_0^{+\infty} e^{-y} dy$$

$$= f \frac{1}{2a} \left(-e^{-y}\right)_0^{+\infty}$$

$$= f \frac{1}{2a} \left(-0 - (-1)\right) = 1$$ for NORMALISATION

$$f \frac{1}{2a} = 1 \quad \Rightarrow \quad f = 2a$$

OK

3d) The prefactor in 2D is "v" (see previous derivation) while in 3D is "v^2" because of the infinitesimal spherical surface we find deriving the probability.

$$p(v) = p_x(v_x) \, p_y(v_y) \, p_z(v_z) \underbrace{dv_x \, dv_y \, dv_z}_{4\pi v^2 dv}$$

where $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$.

Again the volume in probability space is :

$$\frac{4}{3}\pi (v+dv)^3 - \frac{4}{3}\pi v^3 = \frac{4}{3}\pi \left(\cancel{v^3} + 3v^2 dv + \underbrace{\cancel{3 dv^2}}_{2^{nd}\text{ order}} + \underbrace{\cancel{dv^3}}_{3^{rd}\text{ order}} - \cancel{v^3}\right) = 4\pi v^2 dv$$

3e) Relation particle speed & energy $E = \frac{1}{2} m v^2$

show that :

$$P(E)\,dE = \frac{2a}{m} \exp\left[-\frac{2a}{m}E\right] dE$$

We have $P(v)\,dv = 2a\, v \exp\left[-a v^2\right] dv$

$$v = \sqrt{\frac{2}{m}}\sqrt{E} \quad , \quad dv = \sqrt{\frac{1}{2m}}\frac{1}{\sqrt{E}}\,dE$$

$$p(v)\,dv \longrightarrow p(E)\,dE = 2a\sqrt{\frac{\cancel{2}}{m}}\cancel{\sqrt{E}} \exp\left[-\frac{2a}{m}E\right]\frac{1}{\sqrt{\cancel{2}m}}\frac{1}{\cancel{\sqrt{E}}}\,dE$$

$$= \frac{2a}{m}\exp\left[-\frac{2a}{m}E\right] dE$$

OK

3f) Average energy of the particles in 2D

$$\langle E\rangle = \int_0^{+\infty} E\, p(E)\, dE$$

NB $E = \frac{1}{2} m v^2$ E can NOT be NEGATIVE $\Rightarrow \int_0^{+\infty}$

$$= \frac{2a}{m}\int_0^{+\infty} E \exp\left[-\frac{2a}{m}E\right] dE$$

$$y = \frac{2a}{m}E \quad , \quad E = \frac{m}{2a}y \quad , \quad dE = \frac{m}{2a}dy$$

$$\langle E\rangle = \frac{2a}{m}\int_0^{+\infty}\frac{m}{2a}\, y\, e^{-y}\,\frac{m}{2a}\,dy$$

$$= \frac{m}{2a}\underbrace{\int_0^{+\infty} y\, e^{-y}\, dy}_{1}$$

$$= \frac{m}{2a}$$

OK

3g) Equipartition theorem : $\langle E \rangle = \underset{\text{quadratic degrees of freedom}}{\text{dof}} \cdot \frac{1}{2} k_B T$

In 2D $\langle E \rangle = k_B T$.

$$k_B T = \frac{m}{2a} \Rightarrow a = \frac{m}{2 k_B T}$$

3h) Calculate the probability that the energy of a particle randomly chosen is less than $k_B T$.

We have the probability density. To calculate the probability for a particle to be within a certain energy, we have to integrate $p(E)$ up to the energy considered.

$$P(E < k_B T) = \int_0^{k_B T} p(E)\, dE$$

$$= \frac{2a}{m} \int_0^{k_B T} \exp\left[-\frac{2a}{m} E\right] dE$$

$$= \frac{\cancel{2}}{\cancel{m}} \frac{\cancel{m}}{\cancel{2} k_B T} \int_0^{k_B T} \exp\left[-\frac{\cancel{2}}{\cancel{m}} \frac{\cancel{m}}{\cancel{2} k_B T} E\right] dE$$

$$= \frac{1}{k_B T} \int_0^{k_B T} \exp\left[-\frac{1}{k_B T} E\right] dE$$

$$= \frac{1}{\cancel{k_B T}} (-\cancel{k_B T}) \left[e^{-\frac{1}{k_B T} E} \right]_0^{k_B T}$$

$$= -\left(e^{-1} - e^{0}\right) = 1 - \frac{1}{e} \simeq 63\ \%$$

OK